Key Kinematics Concepts

- Change = slope = derivative

  $$v_x = \frac{dx}{dt} \quad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

  Velocity is the slope of position vs t acceleration is the slope of velocity vs. t and the curvature of position vs t.
- Even in simple 1D motion, you must understand the vector nature of these quantities
- Initial conditions
- All formulas have assumptions

Circular Motion Summary

- Motion is a circle with constant speed and radius is accelerated motion.
- The velocity is constant in magnitude but changes direction. It points tangentially.
- The acceleration is constant in magnitude but changes direction. It points radially inward.
- The magnitude of the acceleration is given by: $|\vec{a}| = \frac{v^2}{R}$

Newton s Three Laws

- If $\vec{v}$ is constant, then $\sum \vec{F}$must be zer0 and if $\sum \vec{F} = 0$, then $\vec{v}$ must be constant.
- $\sum \vec{F} = m\vec{a}$
- Force due to object A on object B is always exactly equal in magnitude and always exactly opposite in direction to the force due to object B on object A.

Some Advice

- Your instincts are often wrong. Be careful!
- $\sum \vec{F} = m\vec{a}$ Is your friend. Trust what it tells you.
- Problem Solving Tool: (Revised) Free-Body Checklist
  - Calculate components: $\sum \vec{F}_x = m\vec{a}_x \quad \sum \vec{F}_y = m\vec{a}_y$

Properties of Friction - Magnitude

- Not slipping: The magnitude of the friction force can only be calculated from $\sum \vec{F} = m\vec{a}$.

  However, it has a maximum value of $|f| \leq \mu_s N$
- Just about to slip: $|f| \leq \mu_s N$ where N is the Normal force and μ_s is the coefficient of static friction which is a constant that depends on the surfaces
- Slipping: $|f| \leq \mu_s N$ where N is the Normal force and $\mu_k$is the coefficient of kinetic friction which is a constant that depends on the surfaces
- Note: $\mu_s > \mu_k$

Properties of Spring Force

- The direction is always unambiguous!
  - In for stretched spring, out for compressed spring.
- The magnitude is always unambiguous!
  - $|F| = k(e - e_0)$
- Two possibilities for confusion
  - Double negative: using F = -kx where it does not belong
  - Forgetting the "unstretched length", e_0

Work done by a Force

- Not a vector quantity (but vector concepts needed to calculate its value).
- Depends on both the direction of the force and the direction for the motion
- Four ways of saying the same thing
  - Force times component of motion along the force.
  - Distance times the component of force along the motion.
  - $W = \sum |F||d|\cos(\theta)\ where\ \theta\ is\ the\ angle\ between\ F\ and\ d\,.$
  - $W = \int \vec{F} \cdot \overrightarrow{ds}\ where\ the\ svector\ is\ along\ the\ path$

Checklist to use Work Energy

- Clearly define what is "inside" your system.
- Clearly define the initial and final conditions, which include the location and speed of all objects(s)
- Think carefully about all forces acting an all objects
- <u>**All**</u> forces must be considered in the Work term <u>**or**</u> in the Potential Energy term, <u>**but never in both.**</u>

  $$W = \Delta E = E_{Final} - E_{Initial}$$
  $$= (KE_{Final} + PE_{Final}) - (KE_{Initial} + PE_{Initial})$$

Work Energy Summary

- $W = \Delta E = E_F - E_I \quad E = PE + KE \quad KE = \frac{1}{2}mv^2$
- $PE_{gravity} = mgy \quad PE_{spring} = +\frac{1}{2}k(L - l_0)^2$
- $W = \int \vec{F} \cdot d\vec{s} \quad |W| = |F||ds|\cos(\theta)$
- Every force foes in the work term or in the PE
- Minima and maxima of the PE correspond to F = 0, which are equilibrium points. PE minima are stable equilibrium points, maxima are unstable.

Momentum

- Very simple formula: $\vec{p}_{Tot} = \sum(m_i\vec{v}_i)$
  - Note the vector addition!
- Momentum of a system is conserved only if:
  - No net external forces acting on the system.
  - Or, study the system only over a very short time span.

  $$\Delta\vec{p}_{Tot} = \int \vec{F}dt$$

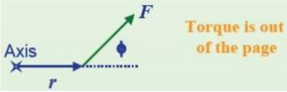
Kinematics Variables

Position	**x**	Angle	**θ**
Velocity	**v**	Angular velocity	**ω**
Acceleration	**a**	Angular acceleration	**α**
Force	**F**	Torque	**τ**
Mass	**M**	Moment of Inertia	**I**
Momentum	**p**	Angular Momentum	**L**

$$\boldsymbol{\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$

Torque

- How do you make something rotate? Very intuitive!
  - Larger force clearly gives more "twist".
  - Force needs to be in the right direction (perpendicular to a line to the axis is ideal).
  - The "twist" is bigger if the force is applied farther away from the axis (bigger lever arm).
- In math-speak: $\vec{\tau} = \vec{r} \text{ x } \vec{F} \quad |\tau| = |r||F|\sin(\varphi)$



Torque Checklist

- Make a careful drawing showing **where** forces act
  - Clearly indicate what axis you are using
  - Clearly indicate whether CW of CCW is positive
- For each force:
  - If force acts at axis or points to or away from axis, τ = 0
  - Draw (imaginary) line from axis to point force acts. If distance and angle are clear from the geometry

    τ = Fsin(θ)
  - Draw (imaginary) line parallel to the force. If distance from axis measured perpendicular to this lice (lever arm) is clear, then the torque is the force times this distance.
- Do not forget CW cersus CCW, it the torque + or –

Right Hand Rules

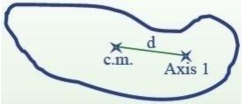
- For angular quantities: θ,ω,τ
  - Curl the fingers of your right hand in the direction of the motion or acceleration of torque and your thumb points in the direction of the vector quantity.
    - The vector direction for "clockwise" quantities is "into the page" and "counterclockwise" is "out of the page"
- Vector cross – products (torque, angular momentum of pint particle) generally A x B
  - Point the fingers of your right hand along the first vector; curl your fingers to point along second vector, your thumb points in the direction of the resulting vector.

Moment of Inertia

- Most easily derived by considering Kinetic Energy
- $I = \sum m_i r_i^2 = \int r^2 dm$
- Some simple cases are given in the textbook.
  - Hoop (all mass at the same radius) $I = MR^2$
  - Solid cylinder or disk $I = MR^2$
  - Rod around end $I = \frac{1}{2}MR^2$
  - Rod around center $I = \frac{1}{3}ML^2$
  - Sphere $I = \frac{2}{5}MR^2$

Parallel Axis Theorum

- Very simple way to find moment of inertia for a large number of strange axis locations.



- $I_1 = I_{c.m.} + Md^2$ where M is the total mass.

Everything you need to know for Linear & Rotational Dynamics

- $\sum \vec{F} = M\vec{a}$
- $\sum \vec{\tau} = I\vec{\alpha}$
  - This is true for **any fixed** axis and for an axis through the center of mass, even if the object moves or accelerates.
- Rolling **without** slipping: $v = R\omega \quad a = R\alpha \quad f \neq \mu N$
  - Friction does NOT do work!
- Rolling **with** slipping: $v \neq R\omega \quad a \neq R\alpha \quad f = \mu N$
  - Friction does work, usually negative.
  - Rarely solvable without using and torque equations!

Kinetic Energy with Rotation

- Adds a new term not a new equation!
- Rotation around and fixed pivot: $KE = \frac{1}{2}I_{pivot}\omega^2$
- Moving and rotating: $KE = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}M_{tot}v_{CM}^2$

Angular Momentum

- Conserved when external torques are zero or when you look over a very short period of time.
  - True for any fixed axis and for the center of mass
- Formula we will use is simple: $\vec{L} = I\vec{\omega}$
  - Vector nature (CW or CCW) is still important
- Point particle: $\vec{L} = \vec{r} \text{ x } \vec{p}$
- Conservation of angular momentum is a separate equation from conservation of linear momentum
- Angular impulse: $\vec{\tau} = \frac{d\vec{L}}{dt} \quad \Delta\vec{L} = \int \vec{\tau}dt$

??????A bicycle wheel *angular position* is given by: **$\theta(t) = (2.0\ rad/s^3)t^3$**
The diameter of the wheel is **0.36 meters**.

a) **Find** the **angle θ**, in radians and in degrees at times **t_1 = 2.0s** and **t_2 = 5.0s**
$\theta(t) = (2.0\ rad/s^3)t^3 \rightarrow \theta_1 = (2.0\ rad/s^3)(2s)^3 = 16\ rad \rightarrow \theta_2 = (2.0\ rad/s^3)(5s)^3 = 250\ rad$
In degrees we have: $16\, rad\left(\frac{360}{2\,\pi\, rad}\right) = 920\, deg$ Or 14,000 degrees

b) **Find** the *distance a particle moves* on the rim during these two time intervals.
The wheel turns a distance **250 rad – 16 rad** = **234 rad**.
The radius is half the diameter and so r = 0.18meters
There for the distance a particle travels during this angle change on the wheel is: **s = rθ = (0.18m)(234rad) = 42 meters.**

c) **Find** the *average angular velocity*, in rad/s and rev/min during these two time intervals.

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250rad - 16rad}{5.0s - 2.0s} = 78\frac{rad}{s}$$

d) **Find** the *instantaneous angular velocity* at these two time intervals.

$$\omega_z = \frac{d}{dt}\theta(t) = \frac{d}{dt}2.0\frac{rad}{s^3}t^3 = 6.0\frac{rad}{s^2}t^2$$

$$\text{At} \rightarrow \mathrm{t} = 5\mathrm{s}: \omega_z = 6.0\frac{rad}{s^2}5.0^2 = 150\frac{rad}{s}$$

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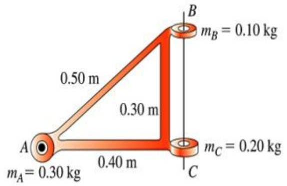
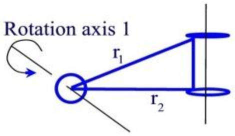
1. What is the **moment of inertia** of this body about an axis through point A, perpendicular to the plane of the figure above? *Consider the struts connecting points A, B and C as massless*.
2. What is the moment of inertia through the axis BC?
3. If the body in figure rotates about an axis through A perpendicular to the plane of the diagram, with angular speed ω = 4.0 rad/s, what is its kinetic energy?



→Moment of Inertia $I = \Sigma m_i r_i^2$ →Recall r is the perpendicular distance to the axis.

Answer a)



→Notice that **mA** is on the axis of rotation where r is zero.
→Only masses **mB** and **mC** need be considered for the moment of inertia of all three masses turning around the axis through mA.
$I = (0.10\ kg)(0.50\ m)^2 + (0.20\ kg)(0.40\ m)^2 = \mathbf{0.057\ kg \cdot m^2}$
$0.057\ kg \cdot m^2$ is them moment of Inertia through ring A.

Answer b)
→The axis of rotation is through **B-C**. Since those masses are at origin r = 0, the perpendicular distance of mass to axis.
→The only mass to consider is mass A and its perpendicular distance is **0.40 m**.
$\mathbf{I} = \Sigma m_i r_i^2 = (0.30\ kg)(0.40\ m)^2 = \mathbf{0.048\ kgm^2}$
Moment of inertia through rings BC

Answer c)
What is the **kinetic energy** of the object if is rotates at **ω = 4.0 rad/s** through ring A?
→$0.057\ kg \cdot m^2$ is them moment of Inertia through ring A.
→$\frac{1}{2} I\omega^2 = \frac{1}{2}(0.057\ kg \cdot m^2)(4.0\ rad/s)^2 = 0.46$ Joules

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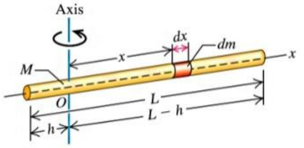
Figure shows a slender rod of mass **M** and length **L**.
??Compute the its moment of inertia about an axis through O, at an arbitrary distance h from one end.



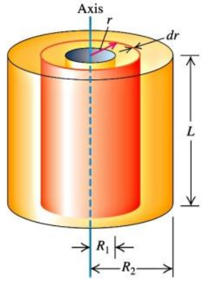
Answer:
→The moment of inertia for any continuous body is given by: $I = \int r^2 dm$
→The next step is to always re-write **dm** as a function of some geometry (unless r is a constant).
→In this case the rod has constant linear density so we write: **dm = λdx**.

$$\int r^2 dm = \int x^2 * \left(\frac{M}{L}dx\right) = \frac{M}{L}\int_{-h}^{L-h} x^2 dx$$

$$= \left[\frac{M}{L}\frac{x^3}{3}\right]_{-h}^{L-h} = \frac{1}{3}M[L^2 - 3Lh + 3h^2$$

$$I = \frac{1}{3}M[L^2 - 3Lh + 3h^2$$

We see if h = 0 the moment of inertia for thin rod reduces to: $I = \frac{1}{3}ML^2$



Find the moment of inertia for a **hollow cylinder** as shown in the figure.

Answer
→Again we use the formula for continuous bodies to find the moment of inertia for the hollow cylinder. $I = \int r^2 dm$
→And again we write dm in terms of the geometry of the object so that r and dm can integrate together.

dm = ρdv = ρ(2πrL dr)

$$I = \int r^2 dm = \int r^2\, \rho 2\pi r L dr = \rho 2\pi r L\int_{R1}^{R2} r^3\, dr = \frac{2\pi\rho L}{4}R_2^4 - R_1^4$$

$$I = \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

The volume however of the solid part of cylinder is given by:
$V = \pi L(R_2^2 - R_1^2)$
This just subtracts out hollow part.
Thus the total mass is given by: $M = \rho\pi L(R_2^2 - R_1^2)$
And so the moment of inertia for the hollow cylinder is given by:

$$I = \frac{1}{2}M(R_2^2 - R_1^2)\ \text{ If } R_1 = 0 \text{ then: } I = \frac{1}{2}MR$$

→A **turbine fan** in a jet engine has a moment of inertia **I = 2.5kg·m2** about its axis of rotation.
→As the turbine is starting up, its angular velocity **ω** as a function of time is: **ω = (40 rad/s3)t2**

a) Find the fan's **angular momentum** as a function of time, and find its value at time **t = 3.0 seconds**
A turbine fan $I = 2.5 kg \cdot m^2$ about its axis of rotation. ω as a function of time is: $\boldsymbol{\omega = (40\ rad/s^3)t^2} \rightarrow L(t) = \boldsymbol{I\omega = (2.5kg \cdot m^2)[(40rad/s^3)t^2]}$
This is the angular momentum (**L**) as a function of t. At **3 s** the **L** would be:
$L(t) = \mathbf{I\omega} = (2.5kg \cdot m^2)[(40rad/s^3)3^2] = \mathbf{900\ kg\ m^2/s}$

b) Find the net torque acting on the fan as a function of time and the torque at **t = 3.0 seconds**→$L(t) = I\omega = (2.5kg \cdot m^2)[(40rad/s^3)t^2]$
Since torque equals the time rate of change of angular momentum we have:

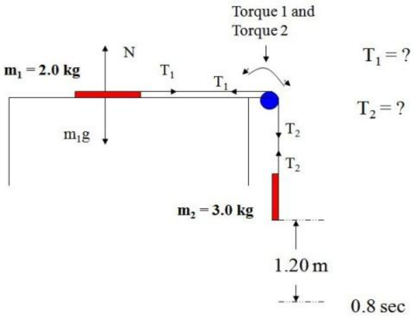
$$\tau = \frac{dL}{dt} = (2.5kg * m^2)\left(40\frac{rad}{s^3}\right)\frac{dt^2}{dt} = (200kg * m^2\frac{rad}{s^3})t$$

The torque at three seconds is:

$$\tau = \left(200kg * m^2\frac{rad}{s^3}\right)3s = 600kg * m^2\frac{rad}{s^2}$$

5.0 kg dumbbell in each extended hand.
He is set in rotation making *1 revolution every second*. $L_i = L_f$

Einstein's I_{E0} without the dumbbells is **$3.0 kg \cdot m^2$** when his *arms are outstretched*…and become I_{Ef} = **$2.2 kg \cdot m^2$** when his hands are at his stomach. $Ii\ \omega_i = If\ \omega_f$
The dumbbells are **1.0 m** from the axis initially and **0.20 m** finally.
Treat dumbbells as particles. $I_i\ \omega_i = I_f\ \omega_f$
→1) Find Einstein's new angular velocity when his hands are pulled into his body. Thus $Ii = I_{prof} + I_{dumbbell} = 3.0\ kg \cdot m^2 + 2(5.0\ kg)(1.0\ m)^2 = 13\ kg \cdot m^2$
$\omega_i = 1\ rev/2.0s = 0.50\ rev/s \rightarrow (0.50\ rev/s)(2\pi\ rad/rev) = 3.14\ rad/s$
The final moment of inertia is given by: $I_{final} = 2.2\ kg \cdot m_2 + 2(5.0)\ kg\ (0.20\ m)^2 = 2.6\ kg \cdot m^2 \rightarrow I_i\ \omega_i = I_f\ \omega_f \rightarrow \frac{12kgm^2}{2.6kgm^2}0.50\frac{rev}{s} = 2.5\frac{rev}{s}$ (2.5 rev/s)(2π rad/rev) = 15.7 rad/s
2) How does this affect kinetic energy?
$K_i = \frac{1}{2} I_i\omega_i^2 = \frac{1}{2}(13kg\ m^2)(3.14rad/s)^2 = 64\ J$
$K_f = \frac{1}{2} I_f\omega_f^2 = \frac{1}{2}(2.6\ kg\ m^2)(15.7\ rad/s)^2 = 320\ J$



→A 2.00 kg textbook rests on a frictionless, horizontal surface.
→A chord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg.
→The system is released from rest, and the books are observed to move 1.20 m in 0.800 sec.
→What is the tension in each part of the chord?
→What is the moment of inertia of the pulley about the rotation axis?

We have three objects that can move.
Thus we apply ΣF and Στ to each object. Although the books have no real torque. Only the pulley does.
Write out the equations of motion (force and torque).
1) $T_1 = ma \rightarrow \Sigma F_{x1} = T_1 = +\ ma \rightarrow$7.50N
2) $T_2 = m_2g - ma \rightarrow \Sigma F_{y2} = T_2 - m_2g = -\ ma \rightarrow$18.2
3) $T_1R - T_2R = I\alpha \rightarrow \Sigma\tau_z = T_1R - T_2R = I\alpha$
Note we have four unknowns, **T_1, T_2, a and I.**
We need the given information to help us solve for tensions in each string.
Use kinematic: $\Delta x = v_{0x}t + \frac{1}{2}a_xt^2 \rightarrow a_x = \frac{2(\Delta x)}{t^2} = \frac{2(1.20m)}{(0.800s)^2} = 3.75\frac{m}{s^2}$
$(T_2\text{-}T_1)R = 0.803N{*}m \rightarrow \alpha = a_1/R = 50\ rad/s^2$, so $I = \tau/\alpha = 0.016\ kg{*}m^2$